

Stochastic Portfolio Theory: Literature Review[☆]

Ali Al-Aradi

Department of Statistical Sciences, University of Toronto, Toronto, Canada

Abstract

A high-level overview of a number of papers in the stochastic portfolio theory literature.

Stochastic Portfolio Theory and Stock Market Equilibrium - Fernholz and Shay (1982)

This paper is one of the seminal works in the area of stochastic portfolio theory (SPT), which is a flexible framework for analyzing portfolio behavior and market structure and describing conditions for market equilibrium. Unlike many of the classical theories concerning market equilibrium such as the capital asset pricing model (CAPM) and its continuous-time counterpart, SPT is a descriptive, rather than a normative, approach to addressing these issues. As such, it relies on a minimal set of assumptions that are readily satisfied in real equity markets. The main focus of this paper is the introduction of some of the central tools used in SPT, primarily the notion of excess growth, along with a characterization of long-term behavior of stocks and equity markets and the conditions for achieving market equilibrium.

The paper begins by introducing a simple lognormal diffusion model to describe stock price behavior, that is $X(t)$, the price of a stock at a time t , satisfies:

$$\frac{dX(t)}{X(t)} = \alpha dt + \sigma dB(t)$$

where α and σ^2 are the expected rate of return and variance of X , respectively and B is a standard Brownian motion. In the SPT framework it is typical to focus on the logarithm of the stock price process instead, whose dynamics are given by the SDE:

$$d \log X(t) = \gamma dt + \sigma dB(t)$$

[☆]This work is based on a project completed for the Winter 2016 session of the Research Topics in Mathematical Finance course taught by Prof. Sebastian Jaimungal at the University of Toronto.

where $\gamma = \alpha - \sigma^2/2$ is the growth rate of X . The motivation for this is that, for analysis of long-term behavior, the parameter γ is more relevant than α since we have that for any $\epsilon > 0$ and large enough values of t

$$\exp(-\epsilon t) < X(t) \cdot \exp(-\gamma t) < \exp(\epsilon t)$$

with probability one, by the strong law of large numbers for Brownian motions.

Next, the paper introduces the notion of a portfolio process, Z , as a vector valued process $A(t) = (A_1(t), \dots, A_n(t))$ where A_i are functions depending on time and the stock prices satisfying $\sum_{i=1}^n A_i = 1$ that represent the proportion of Z invested in X_i . Hence, the evolution of total portfolio value satisfies:

$$\frac{dZ(t)}{Z(t)} = A_1 \cdot \frac{dX_1(t)}{X_1(t)} + \dots + A_n \cdot \frac{dX_n(t)}{X_n(t)}$$

Emphasis is placed on distinguishing between two portfolio types: a *passive* portfolio is one in which the number of shares in each stock remains constant (this is typically referred to as a buy-and-hold portfolio); a *balanced* portfolio is one where the proportions A_i are constant (i.e. a fixed weight portfolio). An important distinction between the two is that the latter is also a lognormal diffusion processes and hence has associated α_Z , σ_Z and γ_Z parameters. Therefore, it is straightforward to compute the growth rate for a balanced portfolio, but computing the same quantity for a passive portfolio can be more involved.

Yet another portfolio type is the *market* portfolio, M , which is the passive portfolio that holds one share in each stock, and hence satisfies:

$$dM = dX_1 + \dots + dX_n$$

As in numerous classical theories, most notable of which is the CAPM, the market portfolio plays an important role in SPT. One way of seeing why the distinction between balanced and passive portfolios is crucial is to note that in the continuous-time CAPM the market portfolio is treated as a balanced portfolio, which leads to the absurd conclusion that all stocks are equivalent.

The next concept introduced is that of excess growth, which is illustrated through a simple two-asset example. Assuming a risky asset X with $\alpha_X = r$ and a riskless asset, $\$,$ with the same rate of return and no volatility, consider a balanced portfolio with π invested in the risky asset. Noticing that the value of this balanced portfolio follows a lognormal diffusion process, its growth rate can be shown to be:

$$\gamma_Z = r + (\pi - \pi^2) \cdot \sigma^2/2$$

The quantity $\gamma_Z^* = (\pi - \pi^2) \cdot \sigma^2/2$ is referred to as the excess growth rate and is strictly positive whenever $0 < \pi < 1$. This indicates that the growth rate of a long-only balanced portfolio exceeds that of its constituents. Furthermore, this quantity is shown to be equal to zero for passive portfolios and can be related to the relative performance of a balanced portfolio and a passive portfolio by noticing that:

$$\frac{Z(t)}{X(t)^\pi \cdot \$ (t)^{(1-\pi)}} = c \cdot \exp(\gamma_Z^* \cdot t)$$

From this relation it can be seen that every time the passive portfolio returns to the weights π and $1 - \pi$ the balanced portfolio will be greater than it by a factor of $\exp(\gamma_Z^* \cdot t)$, which can be viewed as an accrual of excess growth. The intuition here is that rebalancing to fixed weights in the balanced portfolio embeds a buy-low sell-high mechanism which drives this outperformance. The notion of excess growth is then generalized to multi-asset portfolios by defining it as the difference between the Ito intergal and the Stratonovich integral versions of the portfolio process:

$$\gamma_Z^* = \frac{1}{2} \left(\sum_{i=1}^n A_i \sigma_i^2 - \sum_{i,j} (D_j A_i + A_i A_j) \sigma_{ij} \right)$$

where D_j is the partial derivative with respect to $\log X_j$ and σ_{ij} is the covariance between stocks i and j . The two-asset results extend to the multi-asset case: for a balanced portfolio $\gamma_Z^* \geq 0$ if $0 \leq \pi_i \leq 1$ for all i and $\gamma_Z^* = 0$ for any passive portfolio, including the market portfolio.

Finally, the paper discusses conditions that a model of stock market equilibrium must satisfy in terms of excess growth. Namely, the paper derives equations representing the conservation of excess growth in the market:

$$\begin{aligned} \int (D_j A_i + A_i A_j) Z(dA) &= 0 \quad \text{for } i \neq j \\ \int (A_i + D_i A_i - A_i^2) Z(dA) &= 0 \end{aligned}$$

where $Z(dA)$ is a measure on the space of vector valued functions (portfolios). In the case where the stocks follow lognormal processes this leads to the elegant result that the total excess growth generated by all the portfolios in the market is zero:

$$\int \gamma_Z^* Z(dA) = \int \frac{1}{2} \left(\sum_{i=1}^n A_i \sigma_i^2 - \sum_{i,j} (D_j A_i + A_i A_j) \right) Z(dA) = 0$$

An application of this result is to notice that the two equations above take on a particular form when we restrict to balanced portfolios and it is argued that any model which assumes that all investors hold balanced portfolios (such as CAPM) must satisfy these equations. However, it turns out the only in which the conservation equations can be satisfied is for all stocks to be equivalent, which is the same degeneracy that was discussed earlier.

On the Diversity of Equity Markets - Fernholz (1999a)

This paper expands on the work of Fernholz and Shay (1982) by building on the notion of excess growth and introducing the concept of market diversity and entropy as a measure of diversity, and their role in describing stock market equilibrium. Simply put, market diversity refers to the extent with which capital is distributed in the market. Since the weights of the market portfolio are nonnegative and sum up to one diversity is naturally related to the entropy of a probability distribution. The main purpose of the paper is to study the manner in which market diversity evolves, the mechanisms required for maintaining it and the conditions under which it is compatible with market equilibrium as described by CAPM.

First, stocks are assumed to be continuous semimartingales of the form

$$X(t) = X(0) \cdot \exp \left(\int_0^t \gamma(s) ds + \int_0^t \sum_{\nu=1}^n \xi_{\nu}(s) dW_{\nu}(s) \right)$$

where γ is the growth rate process of the stock and ξ_{ν} is the sensitivity of the stock to the ν -th source of uncertainty, W_{ν} , which is a standard Brownian motion. In differential form, the (logarithm of the) stock price evolves according to

$$d \log X(t) = \gamma(t) dt + \sum_{\nu=1}^n \xi_{\nu}(t) dW_{\nu}(t)$$

A market is defined as a family of stocks satisfying $\mathbf{x}\sigma(t)\mathbf{x}' \geq \epsilon\|\mathbf{x}\|^2$ a.s. for some $\epsilon > 0$ and for all $\mathbf{x} \in \mathbb{R}^n$ and $t \in [0, \infty)$, where $\sigma(t) = \xi(t)\xi(t)'$ is defined as the covariance process and whose entries are related to the cross-variation process $\langle \log X_i, \log X_j \rangle$. Portfolios in this market are adapted processes $\pi = (\pi_1, \dots, \pi_n)$ satisfying $\sum_{i=1}^n \pi_i(t) = 1$ a.s. for $t \in [0, \infty)$. It follows that the value of the portfolio π evolves according to the SDE:

$$\frac{dZ_{\pi}(t)}{Z_{\pi}(t)} = \sum_{i=1}^n \pi_i(t) \cdot \frac{dX_i(t)}{X_i(t)}$$

The first main result of the paper is to show that the evolution of a portfolio's value can be expressed in terms of portfolio's growth rate, which in turn can be expressed as the weighted average of asset growth rates plus an excess growth term. That is,

$$Z_{\pi}(t) = Z_{\pi}(0) \cdot \exp \left(\int_0^t \gamma_{\pi}(s) ds + \int_0^t \sum_{\nu=1}^n \pi_i(s) \xi_{i,\nu}(s) dW_{\nu}(s) \right)$$

$$\text{where } \gamma_{\pi}(t) = \sum_{i=1}^n \pi_i(t) \gamma_i(t) + \gamma_{\pi}^*(t)$$

$$\text{and } \gamma_{\pi}^*(t) = \frac{1}{2} \sum_{i=1}^n \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t) \pi_j(t) \sigma_{ij}(t)$$

Notice that the excess growth rate process is half the difference between the weighted average of the asset variances and the portfolio variance. A few results relating the excess growth rate of a portfolio to its weights are presented, most notably the fact that a portfolio has strictly positive excess growth if it has no short sales and positive weights in more than one asset.

The next topic addressed is that of market diversity. Heuristically, a market is diverse if capital is well-spread among its constituent stocks. More precisely, a market is diverse if there exists $\delta > 0$ such that the market portfolio weights

$$\pi_i(t) = \frac{X_i(t)}{X_1(t) + \dots + X_n(t)}$$

satisfy $\pi_i(t) \leq 1 - \delta$ a.s. for $t \in [0, \infty)$ and $i = 1, \dots, n$. In other words, there is an upper bound the amount of capital that can concentrate in any single asset. Further, it is also possible to characterize market diversity in terms of the excess growth rate process of the market portfolio: a market is diverse if there exists $\delta > 0$ such that $\gamma^*(t) \geq \delta$ a.s. for $t \in [0, \infty)$.

The notion of diversity is also related to equilibrium in the following way: equilibrium requires all assets to have the same growth rate, otherwise stocks with higher growth will eventually dominate the market. This is at odds with diversity, and it is shown that if all stocks have the same growth rate the market is not diverse. Interestingly, if all stocks have constant (but possibly unequal) growth rate processes, it is still the case that the market cannot be diverse. This suggests that for diversity to be compatible with stock market equilibrium there must exist a mechanism for capital to be redistributed between companies. This is achieved via dividend payments.

The notion of entropy as a measure of market diversity is introduced next. The market entropy process is defined by

$$S(t) = - \sum_{i=1}^n \pi_i(t) \log \pi_i(t)$$

First, entropy is related to diversity by noting that a market is diverse if and only if its entropy process is bounded away from zero, i.e. there exists $\epsilon > 0$ such that $S(t) \geq \epsilon$ a.s. for $t \in [0, \infty)$. Another interesting application of market entropy is to define the entropy-weighted portfolio, whose weights are given by:

$$\eta_i(t) = \frac{-\pi_i(t) \log \pi_i(t)}{S(t)}$$

It is then shown that the performance of the entropy-weighted portfolio relative to the market portfolio can be decomposed as follows:

$$d \log(Z_\eta(t)/Z_\pi(t)) = d \log S(t) + \frac{\gamma^*(t)}{S(t)} dt$$

That is, the relative performance is decomposed into change in diversity plus a strictly positive term involving excess growth. If the market is diverse this relation implies a long-term outperformance of the entropy-weighted portfolio, in the sense that $\lim_{t \rightarrow \infty} Z_\pi(t)/Z_\eta(t) = 0$ a.s. A modified version of this relationship when dividends are included is given by

$$d \log(\hat{Z}_\eta(t)/\hat{Z}_\pi(t)) = d \log S(t) + \left(\delta_\eta(t) - \delta_\pi(t) + \frac{\gamma^*(t)}{S(t)} \right) dt$$

where \hat{Z} is the total return version of Z and $\delta_\pi(t) = \sum_{i=1}^n \pi_i(t) \delta_i(t)$ is the dividend process associated with the portfolio π .

Finally, the paper relates diversity to stock market equilibrium. It is shown that for a market to be in equilibrium (in the sense that there does not exist a portfolio that dominates the market portfolio and is, hence, preferred by investors) dividend rates for larger stocks must exceed those of the smaller stocks. This is contrasted with the equilibrium described by the CAPM which imposes a stronger set of conditions.

Portfolio Generating Functions - Fernholz (1999b)

This paper further develops the SPT framework by introducing portfolio generating functions, a tool that can be used to construct dynamic portfolios. The properties of these portfolios are discussed, including their potential to dominate the market portfolio or other functionally generated portfolios.

The main contribution of this paper is to introduce the notion of a portfolio generating function. Let μ and Z be the vector of weights and the value of the market portfolio, respectively. Then a positive C^2 function, \mathbf{S} , defined on a neighborhood of the open simplex in \mathbb{R}^n , $\Delta^n = \{x \in \mathbb{R}^n : x_1 + \dots + x_n = 1, 0 < x_i < 1, i = 1, \dots, n\}$, generates a portfolio π if there exists a drift process Θ such that:

$$d \log(Z_\pi(t)/Z(t)) = d \log \mathbf{S}(\mu(t)) + \Theta(t) dt \quad t \in [0, \infty) \quad \text{a.s.}$$

where $Z_\pi(t)$ is the value of the portfolio π at time t . Any positive C^2 function, \mathbf{S} , defined on a neighborhood of Δ^n with $x_i D_i \log \mathbf{S}(x)$ bounded on Δ^n is a portfolio generating function. In particular, \mathbf{S} generates the portfolio π with weights

$$\pi_i(t) = \left(D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t)) \right) \mu_i(t)$$

and drift process

$$\Theta(t) = \frac{-1}{2 \mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij} \mathbf{S}(\mu(t)) \cdot \mu_i(t) \mu_j(t) \tau_{ij}^\mu(t)$$

where $\tau_{ij}^\pi(t)$ is the relative covariance process of π given by:

$$\begin{aligned} \tau_{ij}^\pi(t) &= \sigma_{ij}(t) - \sigma_{i\pi}(t) - \sigma_{j\pi}(t) + \sigma_{\pi\pi}(t) \\ \sigma_{i\pi}(t) &= \sum_{k=1}^n \pi_k(t) \sigma_{ik}(t) \end{aligned}$$

Aside: the relative covariance process is closely related to the cross-variation process of the quotient process relating stock and portfolio, $\langle \log(X_i/Z_\pi), \log(X_j/Z_\pi) \rangle$, while $\sigma_{i\pi}$ is related to the cross-variation process between stock and portfolio values $\langle \log X_i, \log Z_\pi \rangle$.

Several well-known portfolios can be expressed using portfolio generating functions, for example:

1. The market portfolio: $\mathbf{S}(x) = 1, \Theta(t) = 0$
2. The buy-and-hold portfolio: $\mathbf{S}(x) = c_1 x_1 + \dots + c_n x_n, \Theta(t) = 0$
3. The fixed-weight portfolio: $\mathbf{S}(x) = x_1^{p_1} \dots x_n^{p_n}, \Theta(t) = \gamma_\pi^*(t)$

Portfolio generating functions can even be combined (through geometric averaging) to hold their respective portfolios in varying proportions, i.e. $\mathbf{S} = \mathbf{S}_1^{p_1} \cdots \mathbf{S}_n^{p_n}$ generates the portfolio that holds p_i in π_i . Another useful result is the uniqueness of functionally generated portfolios which depends only on the values of the underlying generating function on Δ^n . The paper notes, however, that not all portfolios are functionally generated and provides a characterization for those that are.

Next, the paper discusses conditions under which a strict dominance relationship between a functionally generated portfolio and the market portfolio can be established. Strict dominance of a portfolio η over another portfolio ξ is defined as the existence of a number $t > 0$ such that $\mathbb{P}(Z_\eta(t)/Z_\eta(0) > Z_\xi(t)/Z_\xi(0)) = 1$. This is achieved by first deriving the conditions under which the drift process of a functionally generated portfolio, Θ , is guaranteed to be positive. The condition for dominance is simply for the drift process to be positive and for the process $\mathbf{S}(\mu(t))$ to have a positive lower bound.

One class of portfolio generating functions of particular interest are those that serve as measures of diversity. A measure of diversity is defined to be any positive C^2 function defined on a neighborhood of Δ^n that is symmetric and concave. The symmetry corresponds to the fact that all stocks are treated in a similar way, while the concavity reflects the fact that shifting capital from larger companies to smaller ones increases the measure. The reason these functions are useful is that the conditions for dominance can be related to the notion of a diverse market. Examples of such functions/portfolios include: the entropy function, the diversity-weighted portfolio, the (quadratic version of the) Gini coefficient and others.

Equity Portfolios Generated by Functions of Ranked Market Weights - Fernholz (2001)

This paper extends the concept of functionally generated portfolios to functions of ranked market weights, which allows for the construction of portfolios based on company size. The motivation for this comes from the fact that the original theory assumed that the dimension of the equity market was constant, with no stocks entering or exiting the market. Since this is not the case in real markets, there was a need for a portfolio construction methodology that is not influenced by this phenomenon. The techniques developed in the paper are applied to studying the so-called size effect as well as studying properties of the diversity-weighted portfolio.

Before arriving at the main results, a number of necessary concepts are introduced. First, there is the notion of a rank process, which is similar to the order statistics of a collection of random variables in that, at any given time, the values of a collection of processes is arranged in descending order. In the SPT context, the processes of interest are the market weights, μ_1, \dots, μ_n and the k th rank process is given by

$$\mu_{(k)}(t) = \max_{i_1 < \dots < i_k} \min(\mu_{i_1}(t), \dots, \mu_{i_k}(t))$$

with the corresponding random permutation, p_t , satisfying

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t) \quad \text{and} \quad p_t(k) < p_t(k+1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t)$$

That is, $p_t(k)$ is the index of the stock occupying the k th rank in terms of market weight at time t with ties being assigned to the lowest index. Another important concept is that of the relative rank covariance process defined by

$$\tau_{(ij)} = \tau_{p_t(i)p_t(j)}(t)$$

The final concept is a measure of the amount of time a semimartingale, X , spends near the origin known as the semimartingale local time, $\Lambda(t)$, and defined by

$$\Lambda_X(t) = \frac{1}{2} \left(|X(t)| - |X(0)| - \int_0^t \text{sgn}(X(s)) dX(s) \right)$$

This process can be shown to be nondecreasing and satisfy:

$$\mathbf{1}_{\{0\}}(X(t)) d\Lambda_X(t) = d\Lambda_X(t), \quad t \in [0, T], \quad \text{a.s.}$$

implying that Λ_X only increases when X is at the origin.

The main theorem of the paper involves generating portfolios from functions of ranked market weights. In previous works, it was shown that positive C^2 functions defined on Δ^n could generate portfolios. But since the transformation from market weights to ranked market weights is nondifferentiable, the result (and proof) is slightly modified: any positive C^2 function, \mathbf{S} , defined on a neighborhood of Δ^n with $x_i D_i \log \mathbf{S}(x)$ bounded generates a portfolio π with weights given by

$$\pi_{p_t(k)}(t) = \left(D_k \log \mathbf{S}(\mu_{(\cdot)}(t)) + 1 - \sum_{j=1}^n \mu_{(j)}(t) \cdot D_j \log \mathbf{S}(\mu_{(\cdot)}(t)) \right) \mu_{(k)}(t)$$

and drift process

$$\begin{aligned} \Theta(t) = & \int_0^t \frac{-1}{2 \mathbf{S}(\mu_{(\cdot)}(s))} \sum_{i,j=1}^n D_{ij} \mathbf{S}(\mu_{(\cdot)}(s)) \cdot \mu_{(i)}(s) \mu_{(j)}(s) \tau_{(ij)}^\mu(s) ds \\ & + \frac{1}{2} \int_0^t \sum_{k=1}^{n-1} (\pi_{p_s(k+1)}(s) - \pi_{p_s(k)}(s)) \cdot d\Lambda_{\log \mu_{(k)} - \log \mu_{(k+1)}}(s) \end{aligned}$$

The first application of this result is made in the study of the size effect, which is the empirical observation that small companies have historically outperformed larger companies. The function $\mathbf{S}_L(x_1, \dots, x_n) = x_1 + \dots + x_m$ is shown to generate a large-stock index, ξ , which holds one share in each of the m largest stocks. The return of this index relative to the market is given by the SDE

$$d \log(Z_\xi(t)/Z_\mu(t)) = d \log \mathbf{S}_L(\mu(t)) - \frac{1}{2} \xi_{(m)}(t) \cdot d\Lambda_{\log \mu_{(m)} - \log \mu_{(m+1)}}$$

It is argued that the first term, which represents the relative capitalization of the large index to the market, will be stable and mean reverting. Thus, the relative performance of the large stock index will be driven primarily by the second term which is decreasing a.s. An analogous argument can be made for the small stock index and serves to explain why small stocks outperform large stocks in the long run.

The next application is on the diversity-weighted portfolio, which is generated by the generating function

$$\mathbf{D}_p(x_1, \dots, x_n) = \left(\sum_{i=1}^n x_i^p \right)^{1/p}$$

where p is a parameter between 0 and 1. Note that \mathbf{D}_p is a measure of diversity, and that the diversity-weighted portfolio tilts away from large stocks and towards small stocks. The portfolio generated by this function has weights:

$$\pi_i(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}$$

The paper studies a large-stock version of the diversity-weighted index generated by the function

$$\mathbf{S}(x_1, \dots, x_n) = \left(\sum_{i=1}^m x_i^p \right)^{1/p}$$

which generates a similar portfolio that involves and is normalized using only the largest m stocks. The relative performance of π to ξ is given by

$$\begin{aligned} d \log(Z_\pi(t)/Z_\xi(t)) &= d \log \mathbf{D}_p(\xi_{(1)}(t), \dots, \xi_{(m)}(t)) \\ &\quad + (1-p)\gamma_\pi^*(t) dt + \frac{1}{2}(\xi_{(m)}(t) - \pi_{(m)}(t)) \cdot d\Lambda_{\log \mu_{(m)} - \log \mu_{(m+1)}} \end{aligned}$$

Once again it is argued that the first term on the RHS is stable and mean-reverting over the long run. The second term, which is based on excess growth of the diversity-weighted portfolio, is increasing. The third term is referred to as leakage as it measures the effect of stocks that become too small and are dropped from the large stock index. The performance of the portfolio depends on the relative magnitude of the last two terms, though a historical analysis from 1939 to 1998 shows that the contribution of these last two terms to relative return was 88% and -41.6% respectively.

Diversity and Relative Arbitrage in Equity Markets - Fernholz et al. (2005)

This paper introduces a pair of notions related to diversity, namely weak diversity and asymptotically weak diversity, and derives sufficient conditions under which these market properties hold. Also shown is how weakly diverse markets admit relative arbitrage opportunities, in which certain portfolios outperform the market portfolio over arbitrary time horizons. A discussion of the compatibility of these arbitrage opportunities with the theory of contingent claim valuation is also presented.

The typical SPT setup is used, with diversity defined in the usual manner though the focus is initially on long-only portfolios that take values in the positive part of the simplex, Δ_+^n . Two notions of relative arbitrage for two portfolios π and ρ are introduced:

- An arbitrage opportunity over a fixed, finite-time horizon $[0, T]$ if

$$\mathbb{P}(Z^\pi(T) \geq Z^\rho(T)) = 1 \quad \text{and} \quad \mathbb{P}(Z^\pi(T) > Z^\rho(T)) > 0$$

- A superior long-term growth opportunity if

$$\mathcal{L}^{\pi, \rho} = \liminf_{T \rightarrow \infty} \frac{1}{T} \log \left(\frac{Z^\pi(T)}{Z^\rho(T)} \right) > 0 \quad \text{holds a.s.}$$

Next, the paper introduces markets that are weakly diverse on the time horizon $[0, T]$, defined as a market for which there exists $\delta \in (0, 1)$ such that

$$\frac{1}{T} \int_0^T \mu_{(1)}(t) dt < 1 - \delta \quad \text{almost surely}$$

Heuristically, this says that the average value of the largest market weight over the time horizon is bounded away from 1. There is also the notion of an asymptotically weakly diverse market where the condition above is replaced by:

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mu_{(1)}(t) dt < 1 - \delta \quad \text{almost surely}$$

The intuition here is that the average value of the largest market weight eventually becomes bounded away from 1. There is also the concept of a uniformly weakly diverse market over $[0, T_0]$ in which the market is weakly diverse for every $T \in [T_0, \infty)$. A simple two-asset example is given for a market that is asymptotically weakly diverse but not diverse nor weakly diverse. Also, a set of sufficient conditions for diversity are derived, a special case of which includes the situation where all but the largest stock have non-negative growth rates and where the

largest stock exhibits a log-pole-type singularity as its market weight approaches $1 - \delta$. More specifically the sufficient conditions are:

$$\gamma_{(k)}(t) \geq 0 \geq \gamma_{(1)}(t) \quad \text{for } k = 2, \dots, n \quad , \quad \text{on the event } \left\{ \frac{1}{2} \leq \mu_{(1)}(t) < 1 - \delta \right\}$$

and $\min_{2 \leq k \leq n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\epsilon}{2} \geq \frac{M}{\delta Q(t)}$ where $Q(t) = \log \left(\frac{1 - \delta}{\mu_{(1)}(t)} \right)$

where n is the number of assets in the market, ϵ and δ are the constants appearing in the non-degeneracy and diversity conditions and M is the upper bound on market variance. These conditions imply market diversity on any given finite-time horizon $[0, T]$.

The paper also demonstrates that the diversity-weighted portfolio, $\pi^{(p)}$, represents a relative arbitrage opportunity with respect to the market portfolio over a sufficiently long, but finite, time-horizon if the market is weakly diverse. In particular, it is shown that

$$\text{if } T \geq T_* = \frac{2}{p\epsilon\delta} \cdot \log n \quad \text{then} \quad \mathbb{P}(Z^{\pi^{(p)}}(T) > Z^\mu(T)) = 1$$

This portfolio is also a superior long-term growth opportunity relative to the market if the market is uniformly weakly diverse over $[T_*, \infty)$. Furthermore, it can be shown that, in a weakly diverse market, for arbitrary time-horizons there exist portfolios that consistently outperform or underperform the market portfolio. Such portfolios represent short-term relative arbitrage opportunities. More precisely, if there exists a portfolio satisfying $\mathbb{P} \left(\frac{Z^\pi(T)}{Z^\mu(T)} \geq \beta \right) = 1$ or $\mathbb{P} \left(\frac{Z^\pi(T)}{Z^\mu(T)} \leq \frac{1}{\beta} \right) = 1$, and $\mathbb{P} \left(\int_0^T \tau_{\pi\pi}^\mu(t) dt \geq \eta \right) = 1$ for some $\beta > 0$ and $\eta > 0$ then there exists a second portfolio satisfying $\mathbb{P} (Z^{\hat{\pi}}(T) < Z^\mu(T)) = 1$. A number of examples of constructing portfolios with these properties are presented.

The final topic in the paper deals with the implications of market diversity (and, by extension, the existence of the relative arbitrage opportunities mentioned above) on the theory of contingent claim valuation. It is shown that if a market is weakly diverse, then the stochastic exponential

$$L(t) = \exp \left(- \int_0^t \vartheta'(s) dW(s) - \frac{1}{2} \int_0^t \|\vartheta(s)\|^2 ds \right)$$

$$\text{where } \vartheta = \sigma'(t)(\sigma(t)\sigma'(t))^{-1}[b(t) - r(t)\mathbf{1}]$$

is a local martingale and a supermartingale but is not a martingale. If it were, this would contradict the existence of the relative arbitrage opportunities outlined earlier. Therefore, weakly diverse markets do not allow for the existence of an equivalent probability measure under which discounted stock prices are martingales. Nevertheless, it is shown that such a

market can still be complete and a technique for constructing exact replication strategies is developed. This leads to a familiar expression for the hedging price of a contingent claim:

$$h^Y = \mathbb{E}[Y \cdot L(T)/B(T)]$$

where Y is the claim's payoff and $B(T)$ is the discount factor. One key difference with the usual theory of option pricing is highlighted and that is that put-call parity does not necessarily hold in weakly diverse markets.

Atlas Models of Equity Markets - Banner et al. (2005)

This paper presents a novel way of modeling an equity market, in which drifts and volatilities of different stocks take on different constant values depending on the stock's rank. In its simplest form all but the smallest stock are assigned zero growth rate while the smallest stock has a positive growth rate. This makes the smallest stock responsible for all the growth in the market and is therefore referred to as the Atlas stock. Models of this form are referred to as Atlas models of equity markets. Stock dynamics in a generalized Atlas model are governed by the SDEs for $i = 1, \dots, n$:

$$\begin{aligned} d \log X_i(t) &= \gamma_i(t) dt + \sigma_i(t) dW_i(t) \\ \gamma_i(t) &= \gamma + \sum_{k=1}^n g_k \cdot \mathbb{1}_{\{X_i(t)=X_{p_k}(t)\}} \\ \sigma_i(t) &= \sum_{k=1}^n \sigma_k \cdot \mathbb{1}_{\{X_i(t)=X_{p_k}(t)\}} \end{aligned}$$

where $\gamma + g_k$ and σ_k are the growth rate and volatility of the k th largest stock. This model is also referred to as the first-order model. It is noted that this model admits a unique equivalent martingale measure on any time horizon and, as such, does not admit the relative arbitrage opportunities that are typically studied in SPT.

The first task is to study the properties of solutions to the aforementioned system of SDEs. This starts by defining a collection of sets $\{\mathcal{Q}_i^{(k)}\}$ for $1 \leq k, i \leq n$ to be the polyhedral domains in \mathbb{R}^n which satisfy the property that $y = (y_1, \dots, y_n) \in \mathcal{Q}_k^{(i)}$ implies that y_i is ranked k among y_1, \dots, y_n . Then let Y_i satisfy:

$$dY_i(t) = \left(\gamma + \sum_{k=1}^n g_k \cdot \mathbb{1}_{\mathcal{Q}_k^{(i)}}(Y(t)) \right) dt + \sum_{k=1}^n \sigma_k \cdot \mathbb{1}_{\mathcal{Q}_k^{(i)}}(Y(t)) dW_i(t)$$

which implies that $X_i(t) = e^{Y_i(t)}$, i.e. Y_i is the log-capitalization process of the i th stock. It is shown that the sum of Y_i is equal to

$$\sum_{i=1}^n Y_i(t) = \sum_{i=1}^n Y_i(0) + n\gamma t + \sum_{k=1}^n \sum_{i=1}^n \int_0^t \sigma_k \mathbb{1}_{\mathcal{Q}_k^{(i)}}(Y(s)) dW_i(s)$$

which can be used along with the strong law of large numbers to conclude that this market is coherent, that is

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \mu_i(T) = 0 \quad \text{almost surely} \quad \forall i = 1, \dots, n$$

Another interesting result is that each stock occupies every rank for the same amount of time asymptotically, i.e. for $1 \leq i, k \leq n$ we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\mathcal{Q}_k^{(i)}}(Y(t)) dt = \frac{1}{n} \quad \text{almost surely}$$

In particular, each stock acts as the Atlas stock $(1/n)$ th of the time.

The next question of interest is regarding the dynamics of the ranked log-capitalization processes and the stability of capital distribution. The ranked processes are given by

$$Z_k(t) = \sum_{i=1}^n \mathbb{1}_{\mathcal{Q}_k^{(i)}}(Y(t)) \cdot Y_i(t)$$

whose dynamics are given by

$$dZ_k(t) = \sum_{i=1}^n \mathbb{1}_{\mathcal{Q}_k^{(i)}}(Y(t)) \cdot Y_i(t) + \frac{1}{2} [d\Lambda^{k,k+1}(t) - d\Lambda^{k-1,k}(t)]$$

where $\Lambda^{k,k+1}$ is the local time at the origin of the process $Z_k(t) - Z_{k+1}(t)$. These are used to study the ergodic behavior of the ratios of successively ranked capitalizations:

$$\Xi_k(t) = \log \left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)} \right) = Z_k(t) - Z_{k+1}(t)$$

It is shown that this quantity converges in distribution to an exponential random variable, ξ_k , so that the ratio (without the logarithm) has an asymptotic Pareto distribution. This leads to the ranked relative weights having the following asymptotic distribution:

$$\lim_{t \rightarrow \infty} (\mu_{(1)}(t), \dots, \mu_{(n)}(t)) = (M_1, \dots, M_n) \quad \text{in distribution}$$

where M_i are functions of the ξ_i . Since these random variables have a complex distribution an approximation is made for the limiting vector where all the ξ_i are replaced by their expectations, and the resulting vector is called the certainty-equivalent of M_1, \dots, M_n .

Next, the paper turns to various portfolios and their growth rates in this market model. Of particular interest are the long-term averages of the growth rate and the excess growth rate of the market, equal-weighted and the diversity-weighted portfolios, as well as their restricted counterparts which only allow for trading in large-cap stocks. Also examined are the large-market behaviors of these portfolios when the number of assets tends to infinity. A number of comparisons are made, which depend on the value of the parameters of the model. In some cases the equal-weighted and diversity-weighted portfolio have greater long-term growth rates compared to the market portfolio, with the inability to invest in small stocks not hindering

this outperformance significantly. In other cases, however, the restricted versions of all three portfolios have the same long-term performance. Another interesting portfolio that is examined is the one that always invests in the Atlas stock. It is shown that this portfolio has the highest attainable long-term growth rate, although it is very difficult to implement in practice. Other portfolio analysis includes computing efficient portfolios (in the Markowitz sense) in this equity market model.

Lastly, the paper examines the notion of diversity in Atlas models. It is noted that although the existence of an EMM implies that the market cannot be weakly diverse, i.e. for every T and $\delta \in (0, 1)$ we have

$$\mathbb{P} \left(\frac{1}{T} \int_0^T \mu_{(1)}(t) dt \right) < 1$$

this probability is estimated to be very close to 1. The estimation shows that this type of Atlas market behaves much like a diverse market over short periods (up to 2 years).

Relative Arbitrage in Volatility-stabilized Markets - Fernholz and Karatzas (2005)

The aim of this paper is two-fold: first, to develop criteria for the existence of relative arbitrage opportunities in equity markets in terms of the excess growth rate of the market portfolio. Secondly, to develop a market model in which these criteria hold. This turns out to be what is referred to as volatility-stabilized markets in which the drift rates of stocks are constant but volatilities depend inversely on the stocks' market capitalization rank.

In previous works it was shown that relative arbitrage opportunities exist in weakly diverse markets. In this paper it is shown that, even in the absence of diversity, a market can admit relative arbitrage if the excess growth rate of the market portfolio is sufficiently high. An alternative interpretation is that there is sufficient "intrinsic volatility" in the market, due to the previously established relation between excess growth and asset volatilities. The sufficient condition is that there exists a function, $\Gamma(t)$ that satisfies:

$$\Gamma(t) \leq \int_0^t \gamma_*^\mu(s) ds < \infty \quad \text{a.s.} \quad \forall t \in [0, \infty)$$

If this holds, then an investor can exploit the intrinsic volatility of the market to construct a functionally generated portfolio that outperforms the market over sufficiently long time horizons with weights given by:

$$\pi_i(t) = \frac{c\mu_i(t) - \mu_i(t) \log \mu_i(t)}{c - \sum_{j=1}^n \mu_j(t) \log \mu_j(t)}$$

for some sufficiently large constant c . The relative arbitrage implied by the weak diversity condition is a special case of this condition with $\Gamma(t) = \gamma_* t$ and $\gamma_* = (\epsilon\delta)/2$, where ϵ is the constant appearing in the non-degeneracy condition. Conversely, if $0 < \gamma_* < M$ then the market is weakly diverse with $\delta = \gamma_*/M$ where M is the constant that appears in the total variance upper bound.

An alternative criterion is based on a generalized version of the excess growth rate given by

$$\gamma_*^{\pi,p}(t) = \frac{1}{2} \sum_{i=1}^n (\pi_i(t))^p \cdot \tau_{ii}^\pi(t)$$

Note that $\gamma_*^\pi(t) = \gamma_*^{\pi,1}(t)$. If this quantity for the market portfolio satisfies:

$$\frac{n^{1-p}}{p} \log n + \zeta \leq \int_0^T \gamma_*^{\mu,p}(t) dt < \infty \quad \text{a.s.}$$

for some $p \in (0, 1)$, $T \in (0, \infty)$ and $\zeta \in (0, \infty)$ then a relative arbitrage opportunity relative to the market can be exploited by the portfolio with weights:

$$\pi_i(t) = p \cdot \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p} + (1-p) \cdot \mu_i(t)$$

that is, a p -weighted average of the diversity-weighted portfolio with parameter p and the market portfolio.

The second part of the part focuses on developing a market model for which the above criteria hold. Similar to Atlas models, the parameters of the stock dynamics depend on a stock's market capitalization. However, in these models the volatility of a stock is inversely proportional to its size (hence the notion of stabilization by volatility). The simplest version of such a model is where the stock dynamics are given by:

$$d \log X_i(t) = \frac{1}{\sqrt{\mu_i(t)}} dW_i(t)$$

Notice that the variance of the stock increases as the stock's market weight decreases, reflecting the observation that smaller stocks behave in a more erratic manner compared to larger stocks. An equivalent formulation is given by:

$$dX_i(t) = \frac{1}{2} (X_1(t) + \dots + X_n(t)) dt + \sqrt{X_i(t) (X_1(t) + \dots + X_n(t))} dW_i(t)$$

The excess growth rate of the market portfolio in such models is computed to be $\gamma_*^\mu = \frac{n-1}{2}$, which satisfies the conditions for relative arbitrage mentioned above. Also, discussed are the method for solving the SDE system for this market, the ergodic behavior of stocks and market weights and a hybrid model in which small stocks are given both large variances and large growth rates:

$$d \log X_i(t) = \frac{\alpha}{2\mu_i(t)} dt + \frac{1}{\sqrt{\mu_i(t)}} dW_i(t)$$

A Forecasting Model for Stock Market Diversity - Audrino et al. (2007)

This paper presents a model for forecasting stock market diversity based on generalized tree-structured (GTS) models. These models allow for the inclusion of exogenous explanatory variables along with past values of the variable of interest and the incorporation of multiple market regimes in the modeling of conditional means and volatilities whose thresholds depend on the level of the exogenous variables.

The motivation for forecasting diversity lies in the fact that the random component of the relative performance of functionally generated portfolios, e.g. the diversity-weighted portfolio, are entirely driven by changes in diversity. This can be seen from the decomposition:

$$d \log \left(\frac{Z^\pi(t)}{Z^\mu(t)} \right) = d \log \mathbf{S}(\mu(t)) + \Theta(t) dt$$

Since diversity is bounded and mean-reverting, the (positive) excess growth rate drift dominates the relative performance in the long-run, leading to relative arbitrage opportunities for sufficiently long investment horizons. However, an investor that can predict changes in diversity can exploit this fact to generate additional returns in the short run. Moreover, predicting changes in diversity can help guide an investor's passive-active asset allocation choice as passive strategies appear to fare better in periods of declining diversity and vice versa for active strategies.

The main modeling tool used to forecast market diversity in this paper is the GTS model. Simply put, this is an extension of the well-known GARCH model which allows for multiple conditional mean and volatility regimes, with regime switches being triggered by multivariate thresholds of endogenous and exogenous variables. The main focus is on predicting changes in log-diversity, $D_t = \log \mathbf{D}_p(\mu_1(t), \dots, \mu_n(t))$, where

$$\mathbf{D}_p(\mu_1(t), \dots, \mu_n(t)) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}$$

The most general (nonparametric) description of the model framework for change in log-diversity is as follows:

$$\begin{aligned} \Delta D_t &= D_t - D_{t-1} = \mu_t + \epsilon_t \\ \epsilon_t &= \sqrt{h_t} z_t, \quad \mu_t = g(\Phi_{t-1}), \quad h_t = f(\Phi_{t-1}) \end{aligned}$$

where Φ_t is the information set at t , z_t is an i.i.d sequence of zero mean, unit variance innovations; h_t is the variance process; μ_t is the conditional mean and ϵ satisfies $\mathbb{E}[\epsilon_t | \Phi_{t-1}] = 0$. The information set contains lagged values as well as a vector of relevant exogenous factors

including macroeconomic variables and Fama-French risk factors: $\Phi_t = \{\tilde{D}_{t-1}, h_{t-1}, \mathbf{x}_{t-1}^{\text{ex}}\}$. The GTS model parametrizes the conditional mean and variance, g_θ and f_θ , using parametric threshold functions and a parameter vector θ . More precisely, a partition $\mathcal{P} = \{\mathcal{R}_1, \dots, \mathcal{R}_k\}$ of the state space G of Φ_{t-1} is determined. On each cell, the dynamics of D_t are given by a local AR(1)-GARCH(1,1) model. That is, the overall conditional mean and volatility functions are given by:

$$g_\theta^{\mathcal{P}}(D, \epsilon, h, x^{\text{ex}}) = \sum_{j=1}^k (a_j + b_j \Delta D) \cdot \mathbb{1}_{\{(D, \epsilon, h, x^{\text{ex}}) \in \mathcal{R}_j\}}$$

$$f_\theta^{\mathcal{P}}(D, \epsilon, h, x^{\text{ex}}) = \sum_{j=1}^k (\alpha_{0,j} + \alpha_{1,j} \epsilon^2 + \beta_j h) \cdot \mathbb{1}_{\{(D, \epsilon, h, x^{\text{ex}}) \in \mathcal{R}_j\}}$$

So the sets in the partition correspond to the various regimes and regime thresholds, while the parameter $\theta = (a_j, b_j, \alpha_{0,j}, \alpha_{1,j}, \beta_j; j = 1, \dots, k)$ represents the parameter values for these regimes.

Estimation of the model parameters is done using (pseudo) maximum likelihood. For a given partition \mathcal{P} , the negative log-likelihood function of the model is given by

$$-\ell(\theta; \Phi_2^n) = - \sum_{t=2}^n \log \left[h_t(\theta)^{-1/2} \cdot p_Z \left((\Delta D_t - \mu_t(\theta)) / \sqrt{h_t(\theta)} \right) \right]$$

where p_Z is the density function of the innovations, z_t , and $\Phi_2^n = \{\Phi_2, \dots, \Phi_n\}$. The optimal threshold function (i.e. the optimal choice for the partition \mathcal{P}) is obtained via a tree-structured procedure partial search (the details of which are contained in previous works focusing on the GTS model).

The paper goes on to apply this model to US equity data, using interest rate, inflation and growth variables as exogenous predictors, as well as returns on Fama-French portfolios. The results are discussed and compared to the output of a regular AR(1)-GARCH(1,1) model as a benchmark. Also discussed are the properties of the various regimes that are estimated as part of the GTS-GARCH model. In-sample fits as well as out-of-sample forecasting performances are also compared.

Finally, a trading strategy based on predicting the direction of diversity is presented. The strategy comprises of investing in the diversity-weighted index when a decrease in diversity is forecasted and investing in the regular cap-weighted index when an increase is forecasted (if no change is forecasted the investor remains in their current portfolio). It is shown that this simple strategy delivers improved risk-adjusted performance relative to the market portfolio

net of transaction costs. This outperformance occurs despite consisting of a switch between two highly correlated portfolios.

Stochastic Portfolio Theory Optimization and the Origin of Rule-Based Investing - Oderda (2015)

This paper uses the SPT framework to explain the superior risk-adjusted performance of rule-based, non cap-weighted investment strategies relative to the market portfolio. These allocation strategies include risk-focused approaches (e.g. the minimum variance portfolio and risk parity portfolio), fundamental-focused approaches (e.g. fundamental indexation and quantitative value investing) and equal-weight portfolios. Most studies attribute the outperformance to exposure to various style factors, e.g. low-beta, low-vol, size and value. The main contribution of this paper is to explain the outperformance using techniques from SPT by derivation of a five-fund separation theorem. That is, the solution to a utility maximization problem yields an optimal portfolio that holds a proportion in each of five passive rule-based portfolios. Note that this diversification across different rule-based portfolios avoids the problem of concentration or cluster risk.

First, the optimization problem is formulated assuming the usual SPT setup with dividends. The logarithm of relative values of a portfolio π (the control variable) to the market portfolio μ is given by:

$$\begin{aligned} d \log \left(\frac{Z_\pi(t)}{Z_\mu(t)} \right) &= \frac{1}{2} [\pi(t)' \text{diag}(\sigma(t)) - \pi(t)' \sigma(t) \pi(t)] dt \\ &\quad + (\pi(t) - \mu(t))' \cdot \delta(t) dt \\ &\quad + \sum_{i=1}^n \pi_i(t) \cdot d \log \mu_i(t) \end{aligned}$$

Here, the noise term is based on the portfolio-weighted changes to market weights and the drift term is the sum of the portfolio's excess growth rate and dividend rate differential (relative to the market). The optimization problem involves maximizing the portfolio relative log wealth subject to budget and tracking error constraints by maximizing the drift of the process above and noting that the noise term is bounded. This leads to the objective function

$$\begin{aligned} U(\pi(t), \delta(t), \sigma(t), \mu(t)) &= \frac{1}{2} [\pi(t)' \text{diag}(\sigma(t)) - \pi(t)' \sigma(t) \pi(t)] + (\pi(t) - \mu(t))' \cdot \delta(t) \\ &\quad - \lambda_1 (\pi(t)' e - 1) - \lambda_2 [(\pi(t) - \mu(t))' \sigma(t) (\pi(t) - \mu(t)) - \chi^2] \end{aligned}$$

where χ is the portfolio tracking error risk. Notice that the problem is convex since σ is positive definite. The solution to the optimization problem is shown to be

$$\pi(t) = (1 - A)\mu(t) + A\eta(t)$$

where the parameter A depends on the tracking risk constraint and tends to zero as the this constraint becomes more stringent ($\lambda_2 \rightarrow \infty$), concentrating the holdings in the market

portfolio. Furthermore, the portfolio η is shown to be a linear combination of four rule-based portfolios: the high cash flow rate portfolio, the equally weighted portfolio, a risk parity (inverse volatility) portfolio and a global minimum variance portfolio,

$$\eta(t) = A_1(t)e'\sigma^{-1}(t)\delta(t) \cdot \eta_{HCF}(t) + L(\eta_{EW}(t), \eta_{RP}(t)) + A_2(t) \cdot \eta_{GMV}(t)$$

These results are extended to the case with quadratic utility where the investor wishes to maximize the expected relative return:

$$\mathbb{E} \left[\frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{dZ_\mu(t)}{Z_\mu(t)} \right]$$

The relative arithmetic return can be written as:

$$\frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{dZ_\mu(t)}{Z_\mu(t)} = d \log \left(\frac{Z_\pi(t)}{Z_\mu(t)} \right) + \frac{1}{2} \left[\pi(t)'\sigma(t)\pi(t) - \frac{1}{2}\mu(t)'\sigma(t)\mu(t) \right] dt$$

The optimal solution in this case has the form:

$$\pi(t) = \mu(t) + \tilde{A}\tilde{\eta}(t)$$

where $\tilde{\eta}$ is a long/short version of η that satisfies $\tilde{\eta}'e = 0$.

References

- Audrino, F., R. Fernholz, and R. G. Ferretti (2007). A forecasting model for stock market diversity. *Annals of Finance* 3(2), 213–240.
- Banner, A. D., R. Fernholz, and I. Karatzas (2005). Atlas models of equity markets. *The Annals of Applied Probability* 15(4), 2296–2330.
- Fernholz, R. (1999a). On the diversity of equity markets. *Journal of Mathematical Economics* 31(3), 393–417.
- Fernholz, R. (1999b). Portfolio generating functions. In *Quantitative Analysis in Financial Markets: Collected Papers of the New York University Mathematical Finance Seminar*, pp. 344. World Scientific.
- Fernholz, R. (2001). Equity portfolios generated by functions of ranked market weights. *Finance and Stochastics* 5(4), 469–486.
- Fernholz, R. and I. Karatzas (2005). Relative arbitrage in volatility-stabilized markets. *Annals of Finance* 1(2), 149–177.
- Fernholz, R., I. Karatzas, and C. Kardaras (2005). Diversity and relative arbitrage in equity markets. *Finance and Stochastics* 9(1), 1–27.
- Fernholz, R. and B. Shay (1982). Stochastic portfolio theory and stock market equilibrium. *The Journal of Finance* 37(2), 615–624.
- Oderda, G. (2015). Stochastic portfolio theory optimization and the origin of rule-based investing. *Quantitative Finance* 15(8), 1259–1266.